The Capacity Coefficients of Spherical Conductors.

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Maxwell* pointed out that the self-capacity coefficient of a conductor is numerically equal to its charge when its potential is unity and all neighbouring conductors are at zero potential. He considered that the "proper definition of the capacity of a conductor" is to define it as being equal to the self-capacity coefficient. Adopting this definition, we may consider that the self-capacity coefficient is the capacity of the condenser formed by the conductor, on the one side, and, on the other side, all neighbouring conductors connected with the earth. This gives a simple physical meaning to the self-capacity coefficient, and in one or two simple cases it enables us to compute its value. In the case of a spherical conductor, however, we can give an equally simple way of regarding it, which leads to easier methods of computing its value. As a knowledge of the self-capacity coefficients of spheres is essential in certain practical problems, for instance, when computing the electric stress at which a spark will occur between unequal spherical electrodes when the dielectric between them is at a given temperature and pressure, simplified methods of finding their values are useful. It is proved below that the self-capacity coefficient of a spherical conductor equals its radius together with the capacity of the condenser formed between the surface of the sphere and the images in the sphere of all external conductors, including the earth connected in parallel.

Many of the formulæ given by the author in his papers† are connected by very simple relations. For brevity, we shall refer to these papers as X and Y respectively. The approximate formulæ given in Y for spherical condensers can be usefully employed for computing the capacity coefficients for external spheres, and, conversely, we can use the tables given by Kelvin‡ and in X, p. 529, for computing the values of the capacities of spherical condensers.

The relations show that in many cases the values of the coefficients and the capacities can be written down almost at once without even using logarithmic tables. We shall also show that the theorem can be applied to simplifying the problem of finding the capacity between a sphere and large

* 'Electricity and Magnetism,' vol. 1, § 87.

† 'Roy. Soc. Proc.,' vol. 82, p. 524 ; vol. 94, p. 206.

‡ Kelvin, Reprint, p. 96.

distant conductors or of finding the capacity between a small sphere placed in a large cavity of given shape in a conductor.

General Theorem.

If a conducting sphere A (fig. 1) be in the presence of a conductor B, and

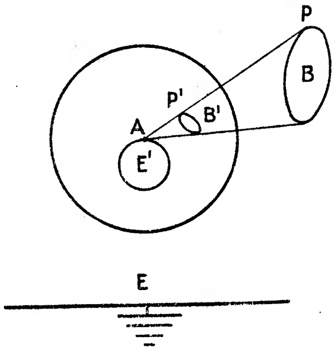

FIG. 1.—AP'. AP $= a^2$. B' and E' are the images of B and the Earth.

if the earth E be at a considerable distance away from the sphere, then the self-capacity coefficient k_{11} of A is given by

$$k_{11} = a + C, \tag{1}$$

where a is the radius of the sphere A, and C is the capacity between the image B' of B in the sphere and the surface of the sphere.

To prove this, let E be the earth, supposed an infinite plane, and let B and E be joined in parallel. The potential of B will be zero. Let a charge q_1 be given to A. There will be induced charges on B and E. Let B' and E' be the inverse surfaces of B and E with regard to the sphere. Then, by the theory of inversion*, a certain charge q' on B' and E' will, when B and E are removed, produce with q_1 the same potential v_1 at the surface of the sphere A as the actual distribution does. If C be the capacity between B' and E' in parallel and the surface of the sphere A, we have

$$q' = C(o - v_1).$$

The quantity of electricity on the inside of the surface of A will be $-q'$. Therefore there will be a charge $q_1 + q'$ on the outer surface, and since B and E are removed, we have the sphere A in infinite space, and so

$$q_1 + q' = av_1,$$

for the capacity of the sphere is a.

Hence $$q_1 = (a + C)\, v_1.$$

But since v_2 is zero we have $q_1 = k_{11}v_1$, and hence

$$k_{11} = a + C. \tag{1A}$$

* J. J. Thomson, 'Electricity and Magnetism,' 4th ed., p. 180.

If the earth E be at a considerable distance away, the sphere E′ will be very minute. Its capacity will, therefore, be negligibly small. Hence, in this case, C is simply the capacity between B′ and A, which proves the theorem. It is obvious that this theorem applies when there are any number of conductors like B in the neighbourhood of the sphere. The condenser C is formed on the one side by all their images connected in parallel, and on the other, by the surface of the sphere. We see at once that the self-capacity coefficient of a sphere is always greater than its radius, and that it is very large when any of the conductors are close to it.

When B forms a continuous conductor entirely enclosing A, we know (Y, p. 208) that the capacity between A and B $= k_{11} = -k_{12}$. Thus, if C_1 be the capacity between a sphere and a conductor entirely surrounding it, and C_2 be the capacity between the surface of the sphere and the image (or inverse) of the surrounding surface in the sphere, we always have

$$C_1 = a + C_2. \tag{2}$$

The Capacity Coefficients of Spheres.

In the following investigation it is found convenient to denote the capacity coefficients k_{11}, k_{22}, and k_{12} of a system of two spheres by $f_{11}(a, b, c)$, $f_{22}(a, b, c)$, and $f_{12}(a, b, c)$ respectively, where a and b are the radii of the spheres and c is the distance between their centres, c being greater than $a+b$. We shall also denote the capacity of a condenser formed by an outer sphere of radius a_1, and an inner sphere of radius b_1 by $F(a_1, b_1, c_1)$, where c_1 is the distance between their centres. It is to be noticed that a_1 must be greater than b_1+c_1, but b_1 can be greater or less than c_1. We also have $F(a_1, b_1, 0) = a_1b_1/(a_1-b_1)$.

Kelvin's* and Maxwell's† formulæ for the self-capacity coefficients of a system of two spheres can be readily proved by (1) and (2). We shall give the proof in full as the steps give relations useful in computation. In order to bring out clearly the relations between the various systems of condensers considered in this theorem, we shall use the functions (α, β, ω) and $(\alpha_1, \beta_1, \omega_1)$ used in X and Y. Their definitions are as follows:—

External Spheres.

$$\left.\begin{aligned} \cosh\alpha &= \frac{c^2+a^2-b^2}{2ca} \\ \cosh\beta &= \frac{c^2+b^2-a^2}{2bc} \\ \cosh\omega &= \frac{c^2-a^2-b^2}{2ab} \end{aligned}\right\}, \tag{I}$$

Internal Spheres.

$$\left.\begin{aligned} \cosh\alpha_1 &= \frac{a_1^2+c_1^2-b_1^2}{2c_1a_1} \\ \cosh\beta_1 &= \frac{a_1^2-b_1^2-c_1^2}{2b_1c_1} \\ \cosh\omega_1 &= \frac{a_1^2+b_1^2-c_1^2}{2a_1b_1} \end{aligned}\right\}. \tag{II}$$

* Kelvin, Reprint, p. 93.

† 'Electricity and Magnetism,' vol. 1, § 173.

We also have,

External Spheres.

$$\left.\begin{aligned} \omega &= \alpha+\beta \\ \sinh\alpha &= \frac{r}{a},\quad \sinh\beta = \frac{r}{b} \\ \sinh\omega &= \frac{cr}{ab} \end{aligned}\right\}, \quad \text{(III)}$$

We also have,

Internal Spheres.

$$\left.\begin{aligned} \omega_1 &= \beta_1-\alpha_1 \\ \sinh\alpha_1 &= \frac{r_1}{a_1},\quad \sinh\beta_1 = \frac{r_1}{b_1} \\ \text{and}\quad \sinh\omega_1 &= \frac{c_1 r_1}{a_1 b_1} \end{aligned}\right\}, \quad \text{(IV)}$$

where r and r_1 are the radii of the orthogonal spheres.

Let us consider two spheres of radii a and b respectively (fig. 2). Let B_1 be the image of B in the sphere A. Then by (1)

$$f_{11}(a, b, c) = a+F(a_1, b_1, c_1), \tag{3}$$

where*
$$a_1 = a, \qquad b_1 = \frac{a^2}{c^2-b^2}\,b, \qquad \text{and} \qquad c_1 = \frac{a^2}{c^2-b^2}\,c. \tag{V}$$

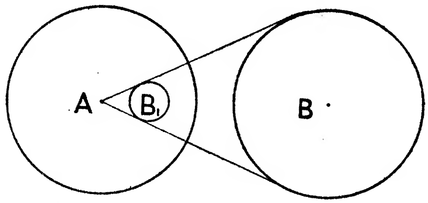

Fig. 2.

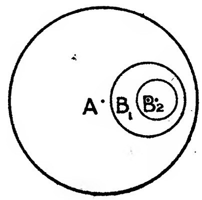

Fig. 3.

Fig. 2.—$k_{11} = a+$the capacity between A and B_1.

Fig. 3.—The capacity between A and B_1 = the radius of B_1 plus the capacity between B_1 and B_2.

Let us now invert the sphere A with respect to the sphere B_1, and let the image be B_2 (fig. 3), we get by (2)

$$F(a_1, b_1, c_1) = b_1+F(a_2, b_2, c_2), \tag{4}$$

where
$$a_2 = b_1, \qquad b_2 = \frac{b_1^2}{a_1^2-c_1^2}\,a_1, \qquad \text{and} \qquad c_2 = \frac{b_1^2}{a_1^2-c_1^2}\,c_1. \tag{VI}$$

We then invert B_1 with respect to B_2 and so on. Thus, by adding up the equations (3), (4) ... we get

$$f_{11}(a, b, c) = a+b_1+b_2+\ldots, \tag{5}$$

and by adding up the same equations, beginning with (4) we get

$$F(a_1, b_1, c_1) = b_1+b_2+b_3+\ldots. \tag{6}$$

Substituting in (II) for a_1, b_1 and c_1, their values from (V), we get

$$\cosh\omega_1 = \frac{a_1^2+b_1^2-c_1^2}{2a_1b_1} = \frac{a^2+a^4b^2/(c^2-b^2)^2-a^4c^2/(c^2-b^2)^2}{2a^3b/(c^2-b^2)}$$
$$= \frac{c^2-a^2-b^2}{2ab} = \cosh\omega.$$

* J. J. Thomson, *ibid.*, p. 177.

Hence $\omega_1 = \omega$. Similarly $\alpha_1 = \alpha$, and thus since $\beta_1 = \omega_1 + \alpha_1$, we get $\beta_1 = \alpha + \omega$. We also have $\sinh \alpha = r/a = \sinh \alpha_1 = r_1/a_1$, and thus $r = r_1$. In the same way we find that $\omega_2 = \omega_1$, $\alpha_2 = \beta_1$, and therefore,

$$\beta_2 = \alpha_2 + \omega_2 = \alpha + 2\omega.$$

Hence also $r_1 = r_2$, etc.

Therefore
$$b_n = \frac{r_n}{\sinh \beta_n} = \frac{r}{\sinh(\alpha + n\omega)}.$$

Hence substituting in (5) and (6) we get

$$f_{11}(a, b, c) = \sum_0^\infty \frac{r}{\sinh(\alpha + n\omega)}, \tag{7}$$

and
$$\mathrm{F}(a_1, b_1, c_1) = \sum_1^\infty \frac{r_1}{\sinh(\alpha_1 + n\omega_1)}$$

$$= \sum_0^\infty \frac{r_1}{\sinh(\beta_1 + n\omega_1)}. \tag{8}$$

Formula (7) is Maxwell's modification of Kelvin's formula,* and (8) agrees with (4) and (5) of Y.

Since we have
$$f_{11}(ma, mb, mc) = mf_{11}(a, b, c), \tag{9}$$

and
$$\mathrm{F}(ma, mb, mc) = m\mathrm{F}(a, b, c), \tag{10}$$

the equations (3) and (4) may be written in the form

$$f_{11}(a, b, c) = a + \frac{a^2}{c^2 - b^2}\mathrm{F}\left(\frac{c^2 - b^2}{a}, b, c\right), \tag{11}$$

$$= a + \frac{a^2}{c^2 - b^2}\mathrm{F}(a_1, b_1, c_1).$$

Note that a_1, b_1, and c_1 in this formula differ from the a_1, b_1, and c_1 in formula (3).

We have also,
$$\mathrm{F}(a_1, b_1, c_1) = b_1 + \frac{b_1^2}{a_1^2 - c_1^2}\mathrm{F}\left(\frac{a_1^2 - c_1^2}{b_1}, a_1, c_1\right) \tag{12}$$

$$= b_1 + \frac{b_1^2}{a_1^2 - c_1^2}\mathrm{F}(a_2, b_2, c_2).$$

In general we can write, when n is greater than unity

$$\mathrm{F}(a_{n-1}, b_{n-1}, c_{n-1}) = \frac{r \sinh(\alpha + \overline{n-2}\,\omega)}{\sinh \alpha \sinh \beta} + \frac{\sinh(\alpha + \overline{n-2}\,\omega)}{\sinh(\alpha + n\omega)}\mathrm{F}(a_n, b_n, c_n), \tag{13}$$

where
$$a_n = \frac{r \sinh(\alpha + n\omega)}{\sinh \alpha \sinh \beta}; \qquad b_n = \frac{r \sinh(\alpha + \overline{n-1}\,\omega)}{\sinh \alpha \sinh \beta};$$

and
$$c_n = c = \frac{r \sinh \omega}{\sinh \alpha \sinh \beta}. \tag{VII}$$

* Kelvin, Reprint, p. 93.

From formula (28) given below, it will be seen that when $c_n^2/(a_n-b_n)^2$ can be neglected compared with unity, we can write

$$\mathrm{F}(a_n, b_n, c_n) = a_n b_n/(a_n-b_n), \tag{14}$$

which is the formula for a concentric spherical condenser. From (VII) also we have

$$\frac{c_n}{a_n-b_n} = \frac{\cosh(\omega/2)}{\cosh\{\alpha+(2n-1)\omega/2\}}.$$

Hence, except when ω is very small, in which case the spheres are very close together and the other simpler formulæ given in X and Y become applicable, we do not need to go to a high value of n before we can use (14).

The equations (11) and (12) can also be written in the inverse form

$$\mathrm{F}(a_1, b_1, c_1) = \frac{a_1^2}{c_1^2-b_1^2} f_{11}\left(\frac{c_1^2-b_1^2}{a_1}, b_1, c_1\right) - a_1 \tag{15}$$

and

$$\mathrm{F}(a_1, b_1, c_1) = \frac{a_1^2}{b_1^2-c_1^2} \mathrm{F}\left(b_1, \frac{b_1^2-c_1^2}{a_1}, c_1\right) - a_1. \tag{16}$$

The Mutual Coefficient.

It is more difficult to prove by this method Kelvin's formula for the mutual coefficient which Maxwell gives in the form

$$f_{12}(a, b, c) = \sum_1^\infty \frac{r}{\sinh n\omega}. \tag{17}$$

It is easy, however, to express its value in terms of the capacity of a spherical condenser. We see from (17) that the mutual coefficient has a constant value for all the families of spheres which have the same r and ω. Let us consider the member of the family for which the radius b of B is infinite (fig. 4). In this case $\beta = 0$, and therefore the radius a_0 of the sphere A_0 in fig. 4 is given by

$$a_0 = r/\sinh \omega.$$

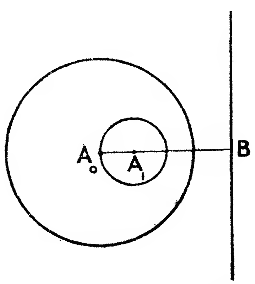

FIG. 4.—$-k_{12}$ equals the radius of A_0 plus the capacity between A_0 and A_1.

We see by the formula for $\cosh \omega_1$ given in (II) that the distance $c-b$ of the centre of the sphere A from the plane is $a_0 \cosh \omega$, that is, $r \coth \omega$.

As a system consisting of a plane and a sphere is the limiting case of a spherical condenser when the outer radius becomes infinite, we have by Y (p. 207), $k_{11} = -k_{12}$, and thus by (1),

$$-k_{12} = a_0 + \mathrm{F}(a_0, r_1, r_1)$$

where r_1 is the radius of the image of the plane in the sphere. Since $r_1 = a_0^2/(2r\coth\omega) = r/\sinh 2\omega$, we have

$$-k_{12} = \frac{r}{\sinh\omega} + \mathrm{F}\left(\frac{r}{\sinh\omega}, \frac{r}{\sinh 2\omega}, \frac{r}{\sinh 2\omega}\right), \tag{18}$$

or

$$-\frac{c}{ab} f_{12}(a, b, c) = 1 + \frac{ab}{c^2-a^2-b^2}\mathrm{F}\left(\frac{c^2-a^2-b^2}{ab}, 1, 1\right). \tag{19}$$

We can also show that

$$f_{12}(a, b, c) = (c_1/c) f_{12}(ax, b/x, c_1), \tag{20}$$

where $c_1^2 = c^2-(a+b)^2+(ax+b/x)^2$, and x is any number.

This follows from (17), because from (I), $\omega_1 = \omega$, and therefore, $\sinh\omega_1 = \sinh\omega$, and so $c_1 r_1 = cr$.

Thus $(c_1/c)\, r_1/\sinh n\omega_1 = r/\sinh n\omega$ and hence (20) follows from (17).

If we put $x = (b/a)^{\frac{1}{2}}$, (20) becomes

$$f_{12}(a, b, c) = (c_1/c) f_{12}(a_1, a_1, c_1), \tag{21}$$

where $a_1^2 = ab$, and $c_1^2 = c^2-(a-b)^2$.

We also have

$$f_{12}(a, b, c) = f_{12}(a_2, a_2, c_2), \tag{22}$$

where $a_2^2 = ab\{1-(a-b)^2/c^2\}$ and $c_2 = c-(a-b)^2/c$.

This follows at once from (21). Hence the mutual coefficient between any two spheres can be found from the Tables* for the mutual coefficients of equal spheres.

Miscellaneous Relations between the Coefficients and Capacities.

If we define the capacity C_0 between two conductors as the ratio of the charge on one of them to the difference of potential between them when they are given equal and opposite charges, we have by X, p. 258,

$$C_0 = \frac{k_{11}k_{22}-k_{12}^2}{k_{11}+k_{22}+2k_{12}}. \tag{23}$$

It readily follows that $k_{11}-C_0$, $k_{22}-C_0$ and $C_0-k_{11}k_{22}/(k_{11}+k_{22})$ are all positive. Hence C_0 lies in value between the smaller of the self-capacity coefficients and $k_{11}k_{22}/(k_{11}+k_{22})$. We may also write (23) in the form

$$-k_{12} = C_0 - \{(k_{11}-C_0)(k_{22}-C_0)\}^{\frac{1}{2}}, \tag{24}$$

* Kelvin, Reprint, p. 96; A. Russell, 'Roy. Soc. Proc.,' vol. 82, p. 529; 'Journ. of the Inst. of Elec. Engin.,' vol. 48, p. 257.

the minus sign being written before the square root as $-k_{12}$ is always less than C_0.

It is easy to show by (7), (8), and (17) that if C_0 be the capacity between the spheres of the system $(b, b, b^2/a)$, then

$$2C_0 = f_{11}(b, b, b^2/a) - f_{12}(b, b, b^2/a) = b + F(b, a, a) \tag{25}$$

Similarly we can show that

$$2C_0 = f_{11}(1, 1, c) - f_{12}(1, 1, c) = -\sqrt{c+2}\, f_{12}(1, 1, \sqrt{c+2}) \tag{26}$$

or

$$2f_0(1, 1, c) = -\sqrt{c+2}\, f_{12}(1, 1, \sqrt{c+2}),$$

and by (19)

$$f_0(1, 1, c) = \tfrac{1}{2} + \frac{1}{2c} F(c, 1, 1), \tag{27}$$

where $f_0(1, 1, c)$ denotes the capacity C_0 between the spheres, 1, 1, of the system $(1, 1, c)$.

The following examples illustrate the practical use of the theorems given above :—

Numerical Examples.

1. A system of two spheres, each 1 cm. in radius and 2 cm. apart.

In this case by (11),

$$k_{11} = f_{11}(1, 1, 4) = 1 + \frac{1}{15} F(15, 1, 4),$$

and by (12)

$$F(15, 1, 4) = 1 + \frac{1}{209} F(209, 15, 4).$$

Keeping the first two terms only in the expansion given in Y, (21) of $F(a, b, c)$ in ascending powers of $c^2/(b-a)^2$, we have

$$F(a, b, c) = \frac{ab}{a-b}\left\{1 + \frac{1}{a/b + b/a + 1}\,\frac{c^2}{(a-b)^2} + \ldots\right\}. \tag{28}$$

Hence

$$F(209, 15, 4) = \frac{209 \times 15}{194}(1 + 0{\cdot}0000284).$$

Therefore

$$k_{11} = 1 + \frac{1}{15} + \frac{1}{194}(1{\cdot}0000284) = 1{\cdot}071\quad 821\quad 4.$$

The value given by Kelvin* is 1·071 82.

If we had assumed that $F(209, 15, 4)$ could be calculated by the ordinary formula for a concentric spherical condenser, namely, $209 \times 15/(209-15)$, the error introduced by this assumption when determining k_{11} would only be unity in the seventh decimal place.

* Kelvin, Reprint, p. 96.

Similarly by (19) we have

$$-4k_{12} = -4f_{12}(1, 1, 4) = 1+\mathrm{F}(14, 1, 1),$$

and by (12),
$$\mathrm{F}(14, 1, 1) = 1+\frac{1}{195}\mathrm{F}(195, 14, 1)$$

and hence by (28)
$$= 1+\frac{14}{181}(1{\cdot}0000020).$$

Therefore
$$-k_{12} = 0{\cdot}269\quad 238\quad 4.$$

Kelvin (*loc. cit. ante*) gives the value 0·269 24.

By (27) we also have

$$f_0(1, 1, 4) = \frac{1}{2}+\frac{1}{8}\mathrm{F}(4, 1, 1),$$

and by (12),
$$\mathrm{F}(4, 1, 1) = 1+\frac{1}{15}\mathrm{F}(15, 4, 1),$$

$$\mathrm{F}(15, 4, 1) = 4+\frac{1}{14}\mathrm{F}(56, 15, 1),$$

and
$$\mathrm{F}(56, 15, 1) = 15+\frac{15}{209}\mathrm{F}(209, 56, 1)$$

$$= 15+\frac{15\times 56}{153}\text{ very approximately.}$$

Hence
$$\mathrm{C}_0 = f_0(1, 1, 4) = \frac{1}{2}+\frac{1}{8}+\frac{1}{30}+\frac{1}{112}+\frac{1}{306}$$

$$= 0{\cdot}670\quad 529\quad 88.$$

Using the values found above, we have

$$\mathrm{C}_0 = \tfrac{1}{2}(k_{11}-k_{12}) = 0{\cdot}670\quad 529\quad 9.$$

2. If we consider the spherical condenser the capacity of which is F (8, 1, 3) and use the inverse formula (15), we get

$$\mathrm{F}(8, 1, 3) = 8f_{11}(1, 1, 3)-8.$$

From Kelvin's Table, $f_{11}(1, 1, 3) = 1{\cdot}146\quad 29$, and thus

$$\mathrm{F}(8, 1, 3) = 1{\cdot}170\quad 32.$$

To check this value, we get by (12)

$$\mathrm{F}(8, 1, 3) = 1+\frac{1}{55}\mathrm{F}(55, 8, 3),$$

and also,
$$\mathrm{F}(55, 8, 3) = 8+\frac{8}{377}\mathrm{F}(377, 55, 3).$$

Hence
$$\mathrm{F}(8, 1, 3) = 1+\frac{8}{55}+\frac{8}{322}(1{\cdot}000\quad 001\quad 4)$$

$$= 1{\cdot}170\quad 299\quad 3.$$

Hence also
$$f_{11}(1, 1, 3) = 1{\cdot}146\quad 287\quad 7.$$

Spheres close together.

When the external spheres are at a distance x apart which is small compared with either radius, the coefficients can be easily computed by the help of Y (16), which gives the capacity of a spherical condenser when the spheres are close together.

As an example of the use of this formula let us find the capacity C between a sphere of radius a which is at a very small distance x from an infinite plane. Making the radius of the outer sphere infinite in Y, (16) and neglecting terms containing squares and higher powers of x/a, we get

$$C = k_{11} = -k_{12} = a\left(1+\frac{x}{3a}\right)\left(\gamma+\tfrac{1}{2}\log\frac{2a}{x}+\frac{x}{9a}\right), \tag{29}$$

where $\gamma =$ Euler's constant $= 0{\cdot}577216$.

Now the capacity C equals a plus the capacity between the sphere and the image of the plane in the sphere. Thus

$$C = a+F\left(a,\ \frac{a^2}{2(a+x)},\ \frac{a^2}{2(a+x)}\right). \tag{30}$$

This equation is exact.

Hence also
$$F(a, c, c)+a, \tag{31}$$
is the capacity between a sphere of a radius a and a plane at a distance x, which equals $a^2/(2c)-a$, from it. For example, when $a = 201$ and $c = 100$, we get $x = 1{\cdot}005$, and thus by (29)

$$F(201, 100, 100)+201 = 201\left(1+\frac{1}{600}\right)\left(0{\cdot}577\ \ 216+\log 20+\frac{1}{1800}\right)$$
$$= 719{\cdot}47,$$

and thus $F(201, 100, 100) = 518{\cdot}47$.

This is correct to the last figure (see Y, p. 214).

The computation of the capacity C between a sphere and a plane is very simple when the distance between them is large compared with the radius of the sphere. For instance, when the radius of the sphere is 10 cm., and the height of its centre above the plane is 100 cm., we get by (30)

$$C = 10+F(10, \tfrac{1}{2}, \tfrac{1}{2})$$
$$= 10+\frac{10\times\frac{1}{2}}{10-\frac{1}{2}} \text{ approx.}$$
$$= 10{\cdot}526\ \ 32.$$

The true value is $10{\cdot}526\ \ 39$.

Approximate Formulæ for the Capacity between a Sphere and a Large Conductor.

If k_{11} be the self-capacity of the sphere, and k_{22} be the self-capacity of all the neighbouring conductors connected in parallel, we have shown above that the capacity C_0 between the sphere and the conductors lies in value between k_{11} and $k_{11} \times k_{22}/(k_{22}+k_{11})$. Hence when k_{22} is large, C_0 is practically equal to k_{11} and thus by (1)

$$C_0 = a + C_1, \tag{32}$$

where C_1 is the capacity between the surface of the sphere and all the images of the conductors in the sphere connected in parallel. If the conductors are distant from the sphere their images will be small and close to the centre of the sphere. In this case C_1 will be very slightly greater than the capacity C' of the images if they were in infinite space. We therefore have

$$C_0 = a + C', \tag{33}$$

approximately. Hence when the capacity C' of the image is known, we can find an approximate value of C_0 at once.

1. Consider a small sphere of radius a placed between two parallel and infinite conducting planes at distances b and c from its centre, where a/b and a/c can be neglected compared with unity. The images of the two planes will obviously be two spheres of radii $a^2/(2b)$ and $a^2/(2c)$ touching one another at the centre of A (fig. 5). The capacity* C' of the images is given by

$$C' = -\frac{a^2}{2(b+c)}\left\{\psi\left(\frac{b}{b+c}\right)+\psi\left(\frac{c}{b+c}\right)+2\gamma\right\}, \tag{34}$$

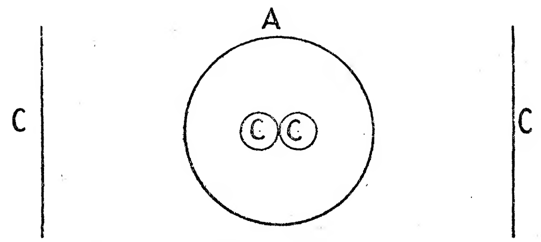

Fig. 5.—The capacity between the sphere A and the two infinite planes equals the radius of the sphere A plus the capacity of the condenser formed by A and the two image spheres touching at its centre.

where $\psi(x)$ is the logarithmic derivate of the gamma function. If we write

* A. Russell, 'Alternating Currents,' vol. 1, p. 243.

$b+c = 2d$, then the capacity C_0 between the sphere and the planes is given by

(1) When $b = c$, $C_0 = a+(a^2/d)\log 2 = a+0{\cdot}693\,(a^2/d)$;*

(2) When $b = 2c$, $C_0 = a+0{\cdot}75\,(a^2/d)\log 3 = a+0{\cdot}819\,(a^2/d)$; and

(3) When $b = 3c$, $C_0 = a+1{\cdot}5\,(a^2/d)\log 2 = a+1{\cdot}040\,(a^2/d)$.

If there had only been one plane at a distance d from the centre of the sphere, this method gives us

$$C_0 = a+0{\cdot}5\,(a^2/d). \tag{35}$$

But by (29) the true value of C_0 in this case is

$$C_0 = a+\mathrm{F}\left(a,\ \frac{a^2}{2d},\ \frac{a^2}{2d}\right).$$

Hence, when $a/d = 1/50$, we have to a seven figure accuracy

$$C_0 = a+\frac{a^2}{2d-a}. \tag{36}$$

In this case, therefore, the error introduced by using (35) instead of (36) is only about the hundredth part of 1per cent.

2. Let us consider the case of an infinite plane conducting sheet with a large hemispherical boss of radius b on it and let us find an approximate value of the capacity C_0 between this sheet and a small sphere of radius a whose centre coincides with the centre of the hemispherical boss. The image of the sheet in the sphere is a hemisphere of radius a^2/b.

Hence $$C' = 2\,(a^2/b)(1-1/\sqrt{3}) = 0{\cdot}845\,(a^2/b).\dagger$$

Hence we find that

$$C_0 = a+0{\cdot}845\,(a^2/b),$$

approximately.

The capacity between the small sphere and the hemispherical boss alone (the rest of the sheet being removed) would be given by

$$C_0 = a+(a^2/b)\left(\frac{1}{2}+\frac{1}{\pi}\right) = a+0{\cdot}818\,(a^2/b).\ddagger$$

If the sphere had been at the centre of a sphere of radius b, its capacity C_0 would be given by

$$C_0 = a+a^2/(b-a) = a+a^2/b \text{ approximately.}$$

We can write down in a similar way the capacity between a small sphere whose centre is on the axis of a long hollow cylindrical conductor, as the

* J. H. Jeans, 'Electricity and Magnetism,' Chap. VIII, ex. 41.

† W. D. Niven, 'London Math. Soc. Proc.,' vol. 8, p. 64; vol. 28, p. 205; H. M. Macdonald, *ibid.*, vol. 26, p. 156; vol. 28, p. 214.

‡ Kelvin, Reprint, p. 178; N. M. Ferrers, 'Quart. Journ.,' vol. 18, p. 97.

image in this case is an anchor ring, the capacity of which is known.* We can also write down at once an approximate value of the capacity between a tetrahedral or a cubical conducting sheet and a small sphere at its centre. In some cases, also, we can assume without appreciable error, that the capacity† of the image can be computed very approximately by the formula $(S/4\pi)^{\frac{1}{2}}$ where S is the surface of the image.

The Lateral Vibrations of Sharply-pointed Bars.

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In a preceding paper,‡ a discussion was given of the lateral vibrations of bars of circular cross-section formed by the revolution of the curve

$$y = Ax^n$$

—when n is between the values zero and unity—about the axis of x. The matter arose in connection with the siliceous deposits found upon a certain type of sponge spicule, as discussed in a joint paper by Prof. Dendy and the present author.§

It is of some interest to obtain a more extended knowledge of the vibrations of solids belonging to this class, with a view to further applications. The phenomena presented change in a curious manner with the value of n, and, in certain respects, could not be foreseen in an elementary way. A discussion of the subject, in numerical terms, for an exponent n between 1 and 2 is very laborious, and in the present paper we confine attention to the case $n = 2$. This is a limiting case, which presents very exceptional features, and gives rise to a period equation of an unusual type. It illustrates clearly, at the same time, the effect of sharpening the ends of the rod beyond the point at which they are conical ($n = 1$). The rod is a free-free bar, symmetrical about its axis, and each half is obtained by the revolution of a portion of a parabola about the tangent at its vertex.

* F. W. Dyson, 'Phil. Trans.,' vol. 184, p. 43.
† A. Russell, 'Journ. Inst. of Elec. Engin.,' vol. 55, p. 12.
‡ 'Roy. Soc. Proc.,' A, vol. 93, p. 506 (1917).
§ 'Roy. Soc. Proc.,' B, vol. 89, p. 573 (1917).